Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

PUBLICATIONS

OF THE

Astronomical Society of the Pacific.

VOL. XIV. SAN FRANCISCO, CALIFORNIA, DECEMBER 1, 1902. NO. 87.

THE LONGITUDINAL ABERRATION OF A PARABOLIC MIRROR.

By W. J. HUSSEY.

In the August-September number of *Popular Astronomy*, Professor WADSWORTH has given an investigation of the longitudinal aberration of a parabolic mirror. He calls attention to the difference between the formula for computing this quantity as given by Dr. DRAPER, viz.,

$$a = \frac{y^2}{4f}, \tag{1}$$

and that given by Mr. WASSELL and others, viz.,

$$a = \frac{y^2}{2f} + \frac{y^4}{16f^3}, \tag{2}$$

where a is the longitudinal aberration measured along the axis from the center of curvature for a zone of radius y of a parabolic mirror whose principal focal length is f.

Professor WADSWORTH concludes that the first of these expressions is correct and that the second is wrong. As a matter of fact, when properly interpreted, both formulas are rigorously correct. They are applicable to parabolic mirrors of any aperture and any focal length. They differ because they correspond to different methods of testing. The first formula assumes that the point-source of light and its image as formed by the zone that is being tested are made to coincide; that is, in testing a mirror having the aberrations computed by this formula the source of light and the eye-piece must move together along the axis of the mirror in passing from zone to zone. The second formula assumes that the point-source of light remains fixed at the center of curvature and that its images corresponding to the various zones do not coincide with it, except in the case of the central zone. In testing a mirror having the aberrations computed by the second formula the eye-piece alone is moved along the axis of the mirror in passing from zone to zone.

The above formulas may be derived easily, without recourse to calculus. Let Figure 1 represent a section of a paraboloid of revolution by a plane containing the axis. Let V be the vertex and F the focus of the parabolic section, and D the center of curvature of the parabola at the vertex. Then $VF = FD = f$. Taking the origin at the vertex and the axis of the parabola as the axis of abscissas, the equation of the parabola is

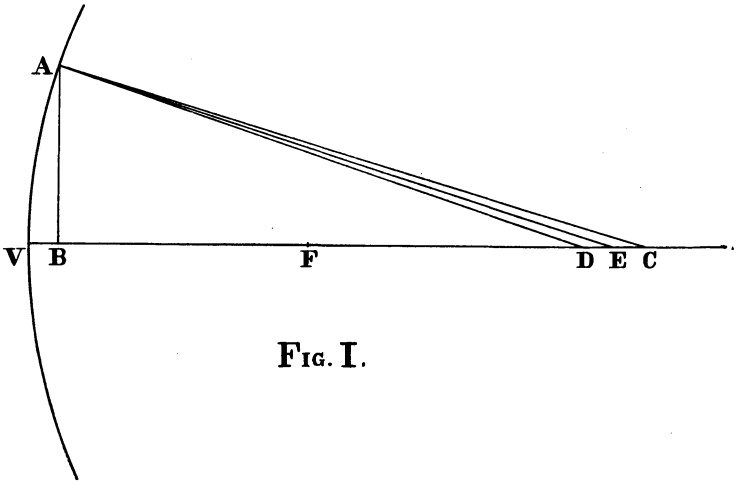

FIG. I.

$$y^2 = 4fx \tag{3}$$

Let A be a point in the parabola at a distance BA from the axis, and AE the normal to the parabola at A. Let the coordinates of A be x' and y'.

CASE I. *Point-source of light and its image coincident.*—Let the point-source of light and its image be placed at E. A ray EA meets the surface of the mirror normally at A and is reflected along the path AE. Moreover, all rays which emanate from E and make an angle equal to VEA with the axis are also reflected in paths which pass through E. E is therefore the image of E as formed by these rays, and the distance DE is the longitudinal aberration of the infinitely narrow zone A of radius $AB = y'$. The subnormal BE of the parabola is constant and equal to $2f$. VD is also equal to $2f$. Consequently, $DE = VB = x' = \frac{y'^2}{4f}$. Or, dropping the accents and writing a in place of DE, we have the required formula as follows:—

$$a = \frac{y^2}{4f}. \qquad (1)$$

CASE II. *Point-source of light fixed at the center of curvature.*—Let the point-source of light be placed at the center of curvature D. A ray DA meets the surface of the mirror at A and is reflected along the path AC, making the angles DAE and EAC equal, these being respectively the angles of incidence and reflection. All rays which emanate from D and make an angle equal to VDA with the axis lie in the surface of a right cone which intersects the paraboloid in a circle of radius $AB = y'$, and they are reflected in rays which all lie in the surface of another right cone whose vertex is at C and whose semi-vertical angle is VCA. C is therefore the image of D formed by the rays which lie in the surfaces of the cones mentioned, and DC is the longitudinal aberration of the infinitely narrow zone A of radius $AB = y'$.

The coordinates of D are $2f$ and 0; those of E are $2f + x'$ and 0; and those of C are $2f + a$ and 0. Having the coordinates of A, D, E, and C, we may write the equations of the lines AD, AE, and AC. They are respectively as follows:—

$$y = \frac{y'}{x' - 2f}x - \frac{x'y'}{x' - 2f} + y', \qquad (4)$$

$$y = -\frac{y'}{2f}x + \frac{x'y'}{2f} + y', \qquad (5)$$

$$y = \frac{y'}{x' - 2f - a}x - \frac{x'y'}{x' - 2f - a} + y'. \qquad (6)$$

In these equations the coefficients of x are the tangents of the angles which these lines make with the axis of abscissas. Now, if m and m' are the tangents of the angles which any two lines make with the axis of abscissas, the angle θ included between these lines is given by the expression,

$$\tan\theta = \frac{m - m'}{1 + m\,m'}. \tag{7}$$

If we substitute in this the values of m and m' from the equations of *DA* and *AE,* and also those of *AE* and *CA,* we shall obtain respectively the tangents of the angles *DAE* and *EAC.* These angles are necessarily equal. We may therefore equate the values of their tangents. Doing this and solving the resulting expression for a, we obtain the following rigorous expression for the longitudinal aberration of a parabolic mirror:—

$$a = 2\,x' + \frac{x'^2}{f}, \tag{8}$$

or, substituting $\frac{y'^2}{4f}$ for x' and dropping the accents,

$$a = \frac{y^2}{2f} + \frac{y^4}{16f^3}. \tag{2}$$

A more general formula for the longitudinal aberration, of which the equation just given is a special case, may be readily derived. It is here considered as

CASE III. *Point-source of light situated upon the axis at a distance d from the vertex.*—The coordinates of the source of light are now d and 0. Let those of its image corresponding to the zone of radius y' be denoted by $h + a$ and 0, where h is the distance from the vertex at which the image is formed by that portion of the mirror which is situated at the vertex, and where a is the aberration required. In this case the aberration is reckoned from a point in the axis at a distance h from the vertex.

The argument and method here are the same as in the preceding case. The equations of the incident ray, the normal, and the reflected ray are respectively as follows:—

$$y = \frac{y'}{x' - d}\,x - \frac{x'\,y'}{x' - d} + y', \tag{9}$$

$$y = -\frac{y'}{2f}\,x + \frac{x'\,y'}{2f} + y', \tag{10}$$

$$y = \frac{y'}{x' - h - a}\,x - \frac{x'\,y'}{x' - h - a} + y'. \tag{11}$$

The expressions for the tangents of the angles included between the first and second, and between the second and third of these lines are formed and equated. When the resulting expression is simplified and x' eliminated through the use of the equation $y'^2 = 4fx'$, and the accents dropped, it reduces to

$$a = \frac{y^2}{2(d-f)} + \frac{y^4}{16f^2(d-f)} + \frac{f(d+h) - dh}{d-f} \,. \qquad (12)$$

Now, from the manner of selecting the distance h, it will be seen that a and y vanish simultaneously. Consequently the constant term of the last equation is equal to zero. From this condition we get the relation between h and d, as follows:—

$$h = \frac{fd}{d-f} = f + \frac{f^2}{d-f}. \qquad (13)$$

At the same time the preceding equation reduces to

$$a = \frac{y^2}{2(d-f)} + \frac{y^4}{16f^2(d-f)}, \qquad (14)$$

which is the complete formula for the longitudinal aberration of an infinitely narrow zone of a parabolic mirror, the source of light being situated anywhere upon the axis.

If the source of light is placed at the center of curvature, $d = 2f$, and $h = 2f$, and the last equation reduces to

$$a = \frac{y^2}{2f} + \frac{y^4}{16f^3}, \qquad (2)$$

as given in Case II.

If the source of light is placed at an infinite distance,—that is, if the mirror is directed to a star,—d becomes infinite and at the same time $h = f$, and $a = 0$. Or, in other words, rays parallel to the axis are brought to a focus from all parts of the mirror, free from aberration, at the principal focus of the mirror.

In the preceding paragraphs it has been assumed that the rays of light are reflected from an infinitely narrow zone of the mirror, the zone being circular and having its center in the axis of the paraboloid. The zones actually used in testing a mirror necessarily have a finite width. In consequence of this the rays of light which are reflected from the surface of such a zone as is used in practice do not all intersect in the same point of the axis. This will be evident from the following concrete example, which we shall regard as

CASE IV. *Point-source of light fixed at the center of curvature D and zone of finite width,* as indicated by *AB, A'B'* of Figure 2.

According to these hypotheses the rays of light which are reflected from *A* and *B* respectively intersect the axis in points *C* and *G,* the aberrations *DC* and *DG* being those given by equation (2). Rays reflected from points between *A* and *B* meet the axis in points between *C* and *G.* The extreme rays *AC* and *BG* intersect in *H,* but since no other rays from the zone under consideration intersect in *H, H* is not an image of *D* as formed by rays from any portion of this zone. On the contrary, the rays which are reflected from that part of the surface from *A* to *B* inclusive, when taken in all possible combinations of two at a time, have points of intersection which completely fill the region *HCG.* The symmetrical portion of the zone *A'B'* likewise furnishes rays which intersect the axis in points between *C* and *G,* and which when taken two at a time have points of intersection which completely fill the region *KCG,* symmetrical with *HCG.*

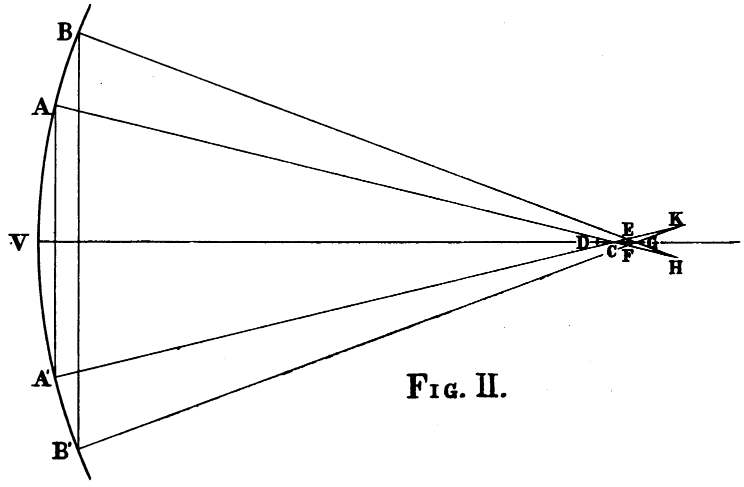

FIG. II.

The rays from *A'* and *B,* and those from *A* and *B'* intersect respectively in the points *E* and *F.* Let the line joining *E* and *F* intersect the axis in L.* Then it is evident that VL is the distance from the vertex at which the cross-section of the rays reflected from the entire circular zone AB, $A'B'$ has its minimum diameter. When the zone is of finite width, as here assumed, there is no point in which all the rays reflected from the zone intersect, and therefore there is no single all-embracing focal point. But in passing the circle of which EF is the diameter the rays are crowded together more closely than elsewhere, so that in this circle we have the nearest approach to the formation of an image of the point D by all the rays which are reflected from the zone of finite width AB, $A'B'$. This circle is called *the circle of least confusion.* We will next determine its position and radius.

Let x', y' be the coordinates of A and x'', y'' those of B. Then the equations of $A'C$ and BG are respectively as follows:—

$$y = -\frac{y'}{x' - 2f - a'}\,x + \frac{x'\,y'}{x' - 2f - a'} - y', \qquad (15)$$

$$y = +\frac{y''}{x'' - 2f - a''}\,x - \frac{x''\,y''}{x'' - 2f - a''} + y'', \qquad (16)$$

where, according to equation (2),

$$a' = \frac{y'^2}{2f} + \frac{y'^4}{16f^3},$$

and

$$a'' = \frac{y''^2}{2f} + \frac{y''^4}{16f^3}.$$

E is the point of intersection of these lines. Its abscissa is the distance of the circle of least confusion from the vertex of the paraboloid, and its ordinate is the radius of this circle. Hence, if we solve equations (15) and (16), to get the coordinates of their point of intersection, we shall have the quantities required. In the unreduced form the resulting expressions are somewhat complicated. If, after forming them, we put

$$y'' + y' = 2z,$$

$$y'' - y' = 2w,$$

---

* The letter L is omitted from the diagram, the space available not being sufficient to give it clearly. Its location is evident from the description of the diagram.

we find that they may be reduced to the following forms:—

$$VL = 2f + \frac{z^2 + 3w^2}{2f} + \frac{z^4 + 6z^2w^2 + 9w^4}{16f^3} + \frac{w^2 e}{fk} + \frac{w^2 g}{4f^3 k}, \quad (17)$$

and,

$$LE = \frac{4w(z^2 - w^2)(4f^2 + z^2 + w^2)}{k}, \quad (18)$$

where, for brevity, we have written

$$k = 32f^4 + 4f^2(z^2 - w^2) + (z^2 - w^2)(z^2 + 3w^2),$$

$$e = -5z^4 - 2z^2w^2 + 7w^4,$$

$$g = -3z^6 - 3z^4w^2 - z^2w^4 + 7w^6.$$

If the aberration is measured from the center of curvature, we have

$$a = VL - VD = VL - 2f$$

as the value of the longitudinal aberration of the circle of least confusion. This is given directly by transposing $2f$ from the second to the first member of equation (17). The last two terms of equation (17) are very small for such mirrors as are used for reflecting telescopes and may be omitted without material error. Neglecting them, we have

$$a = \frac{z^2 + 3w^2}{2f} + \frac{z^4 + 6z^2w^2 + 9w^4}{16f^3}, \quad (19)$$

which is a very approximate expression for the longitudinal aberration of a zone of width $2w$ and mean radius z of a parabolic mirror whose principal focal length is f.

Equation (2) is a special case of the one last given. When an infinitely narrow zone is taken, w may be put equal to zero; the omitted terms vanish, and equation (19) reduces to the form of equation (2). Further, the diameter of the circle of least confusion becomes evanescent along with w, as will be seen from a consideration of equation (18).

CASE V. *Point-source of light situated upon the axis at a distance d from the vertex, the zone being of finite width.*

Let the aberration be measured from a point on the axis at a distance h from the vertex. Then the equations of the reflected rays $A'C$ and BG are like (15) and (16), except that $2f$ is replaced by h and the aberrations a' and a'' now have the complete form of equation (12). The method here is precisely the same as for the preceding case. The results are

$$VL = \frac{fd}{d-f} + \frac{z^2 + 3\,w^2}{2\,(d-f)} + \frac{z^4 + 18\,z^2\,w^2 - 3\,w^4}{16\,f^2\,(d-f)} + \frac{3\,(-z^2\,w^2 + w^4)}{2\,df\,(d-f)}, \tag{20}$$

and

$$LE = \frac{4\,w\,(z^2 - w^2)\,(4f^2 + z^2 + w^2)}{t}, \tag{21}$$

where in the value of VL we have omitted the terms which are insensible in all practical cases. The several omitted terms are those corresponding to the last two terms of equation (17) and like them contain the factor w^2 in their numerators, while their denominators are at lowest of the order f^4.

In the denominator of (21), for brevity we have put

$$t = 16\,f^3\,d + 4\,f\,(z^2 - w^2)\,(3\,f - d) + (z^2 - w^2)\,(z^2 + 3\,w^2).$$

From equation (21) it will be seen that the diameter of the circle of least confusion becomes evanescent along with w.

If the aberration is measured from a point upon the axis at a distance h from the vertex of the paraboloid, we shall have for its value

$$a = VL - h, \tag{22}$$

VL being given, in general, by equation (20) and h by (13). When the aberration is measured from the center of curvature the last equation becomes

$$a = VL - 2f. \tag{23}$$

Equations (1) and (2) may both be obtained from (23) by giving to w, z, and d in VL values corresponding to the conditions imposed in the derivation of (1) and (2). In each case the zone then becomes infinitesimal in width. Hence $w = 0$, and we may put $z = y$. When the light is placed at the foot of the normal, *i. e.* at E in Figure 1,

$$d = x + 2f,$$

and in accordance with these circumstances equations (20) and (23) give

$$a = \frac{y^2}{4f}.$$

When the light is fixed at the center of curvature, *i. e.* at D in Fig. 1,

$$d = 2f,$$

and the same equations give

$$a = \frac{y^2}{2f} + \frac{y^4}{16f^3}.$$

Mt. Hamilton, California, October 22, 1902.

OBSERVATIONS OF THE ECLIPSED MOON, OCTOBER 16, 1902.

By Rose O'Halloran.

The atmosphere here was very favorable to observation during the total phase of the recent lunar eclipse, which occurred on October 16th, in accordance with astronomical forecast. Some minutes before its symmetry was noticeably impaired, the east limb of the Moon was dimmed by the terrestrial shadow, which within half an hour had successively submerged *Grimaldi, Aristarchus, Kepler, Gassendi,* and *Tycho.* Cirrus clouds flitted by during the half-moon phase, and the crescent was formed before the shadowed surface became discernible. A narrow semicircular strip round the limb was first suffused with the glow of refracted sunlight, wreathing the dark terrestrial hemisphere, and large tracts adjacent to *Tycho,* where lunar seas are scarce, were soon included in this interesting illumination. Though familiar with descriptions of our satellite when eclipsed in recent and former years, the color was more vivid than I anticipated, and, telescopically, it was distinctly pink.

At 9:16 p. m. the shadow was over *Mare Crisium* and the crater *Petavius,* but for some minutes after totality the foreshortened limb west of these features shone with a dull white light. This was succeeded by a luminous pink hue that could not be classed as either coppery or chocolate-colored. Probably it was caused by a mingling of white light with unusually red rays from the atmospheric aureole. At first the central portion of the disk seemed to be shrouded from this beacon of totality, but before long the seas were discernible, even without magnifying power. If an observer were uninfluenced by